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On the Calculus of Subjective Probability in Behavioral Economics

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Abstract

In elaborationg upon the recent thought-provoking paper "Subjective probability in behavioral economics and finance: A radical reformulation" by H. Joel Jeffrey and Anthony O. Putman [5], we proceed to specify the calculus of their "probability (uncertainty) appraisals" as *possiblity measures*, i.e., the "radical" reformulation of the usual calculus of subjective probabilities is that of idempotent uncertainty. With possibility measures as quantitative uncertainty for subjective probabilities, we discuss the necessary mariage of possibility measures and Kolmogorov probability measures in a new Bayesian analysis for economic applications.

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1 INTRODUCTION

As stated in the abstract, this paper is about the recent thought-provoking paper [5]. It is thought-provoking because it proposed a "radically new" way to really understand the old notion of subjective probability, from which applications, e.g., in decision-making in social sciences, will be "radically different", say, more "compatible" with what psychological experiments revealed (in behavioral econometrics and finance).

As such, it is important to expose the essentials of [5] to a large audience of econometricians. In fact, by doing so, we accomplish several important tasks, namely specifying the calculus of subjective probabilities as (idempotent) possibility as previously suggested [2], and discussing the necessary mariage of possibility calculus with standard Kolmogorov probability calculus, which is needed in a "Bayesian" framework of statistical inference.

Since the "beginning", we are taught that, in constrast to frequentist probability (Kolmogorov), subjective (or Bayesian) probabilities are in our mind, which are used as a mode of judgement. When using subjective probabilities in applications, such as in auctions (as Bayesian games) or Bayesian statistics, not only we need to manipulate quantitative subjective probability, but also combine subjective probability with frequentist probability (to form Bayesian statistics).

While, the mathematical (language) foundation for quantitative (frequentist) probability is Kolmogorov's axioms (by analogy with measure theory), including additivity, what are the axioms, i.e., the calculus of subjective probability, that the Bayesians use to establish their Bayesian statistical theory?

Can you guess? "Imagine" if the calculus of subjective probabilities is different than Kolmogorov probability calculus! Then how can Bayesian analysis be carried out? Specifically, how to incorporate prior information into a frequentist framework? Well, we know that, as stated again in [5], the standard "view" is that with respect to calculus (of uncertainties), frequentist and subjective probabilities are *two sides of the same coin*, i.e., while their meanings are different, their calculi are the same. This allows Bayesian statistics to exist to "beat" frequentist statistics in several fronts, e.g., in hypothesis testing.

I can't resist to point out an opposite situation: while the meaning of the (intrinsic) uncertainty in quantum mechanics is the same as that of ordinary randomness, their calculi are different, namely, quantum probability is non-commutative whereas Kolmogorov probability is not (but, in fact, quantum probability is simply a non-commutative generalization of Kolmogorov probability).

Should we ask "why it is so?". More specifically, "Why subjective probabilities are additive, or even $\sigma-$additive, just like Kolmogorov probabilities?". Well, texts on subjective probability explain that, by using betting schemes, they are so by existence of Dutch book arguments for these properties.

Note that, although Bertrand Russel once wrote in 1929 "Probability is the most important concept in modern science, especially as nobody has the slighest notion what it means", people equate uncertainty with probability whose calculus is Kolmogorov. Thus, "traditionally", while subjective and objective probabilities are different manifestations of uncertainty, people manipulate them according to the same calculus. After Dennis Lindley attended a seminar at UC Berkeley in 1981 in which both Lotfi Zadeh and Glenn Shaffer talked about their non-additive uncertainty measures (possibility measures, and belief functions, respectively), he wrote [7] claiming that all non-additive uncertainty measures are inadmissible, i.e., the message is "we cannot avoid probability" (where, of course, again, by "probability", we mean additive set functions). It turns out that Lindley's message is not exactly what he claimed! What he did show is that "Using scoring rule approach, an admissible uncertainty measure must be a function of a probability measure". But then, for example, Shaffer's belief functions which are non-additive, are functions of probability measures, and hence admissible in Lindley's sense! A complete response to Lindley's paper was [4].

One more thing about the coexistence of objective uncertainty (say, in von Neumann-Morgenstern utility theory) and subjective uncertainty (say, in Savage's subjective/qualitative probability theory): this is possible in applications since these two different types of uncertainty are forced to obey the same calculus. For a rigorous treatment, see [6], where it reminded the reader that "One warning: when mathematicians use the term probability , they almost always mean a $\sigma-$additive probability measure defined on a $\sigma-$algebra"! It is clear in [6] that uncertainty theories are proposed to be used in modeling behavior of individuals in their decision-making, and as such, e.g., in physics, models of choice must be confirmed by experimental evidence: they were not (See the last Chapter of [6] on "The Experimental Evidence"). Thus, the door was open ever since for non-additive uncertainty measures. It should be emphasized for statisticians that Kolmogorov probability is just one quantitative modeling of one type of uncertainty. There are other types of uncertainty whose modelings might not be "probability", i.e., not additive. Do not equate uncertainty with probability. How to find out reasonable quantitative theories of uncertainty? Well, just follow physics (I mean quantum mechanics)! Quantitative modeling of a type of uncertainty is used to model the behavior of something, and as such, a proposed model must be tested by experiments. The non-commutativity and non-additivity of probability in quantum mechanics, as observed in experiments, led to the establishment of a firm theory of quantum probability. In social sciences, including economics and finance, people make decisions under uncertainty. Thus, it is so clear that any quantitative theory of uncertainty must be validated by *experimental evidence.* The thought-provoking paper under review is precisely in this scientific spirit.

Somewhat clearly (!) that statisticians and econometricians are still not aware of non-additive set functions which were proposed to model various different types of uncertainty we face in everyday decision-making, say in Artificial Intelligence, (not just one type of uncertainty, traditionally attributed to randomness or epistemic uncertainty, as two sides of the same coin, i.e., obeying the calculus of Kolmogorov probability theory), let alone fuzzy set theory of Zadeh (1965, see [10] for a complete update of the theory) from which are founded concepts such as approximate reasoning, soft computing, granular computing, possibility theory (see e.g., [1,3,13–16]). As we will see shortly, the "radical reformulation" of the notion of subjective probability (uncertainty) in [5] is "probability appraisals" which are nothing else than Zadeh's "linguistic variables" [17] in the context of fuzzy set theory. In this context, it looks like we are heading back into the territory of fuzzy set theory which, since 1965, only attracted computing and engineering fileds. Perhaps it will be so since, after all, there are "real" contributions of fuzzy theory to social sciences, including economics and finance.

The purpose of this present paper is multifold: First, we elaborate of the "radical reformulation" of [5] which we believe that it was in a right direction for improving methods in behavioral economics and finance. Next, we place this reformulation completely in the setting of linguistic variables and granular information. Then, we complete [5] by specifying a reasonable calculus of subject probabilities, namely possibility measures (as opposed to probability measures). Finally, we discuss a "Bayesian" analysis in which possibility uncertainty coexists with probability uncertainty.

2 UNCERTAINTY APPRAISALS

In a sense, as opposed to objective uncertainty (quantitatively modeled as frequentist probabilities), by subjective uncertainty we mean the type of uncertainties (e.g., epistemic uncertainty) which cannot be modeled quantitatively by a frequentist approach (e.g., for events which cannot be repeated), it is said that, for such uncertainty, we judge it by using our mind. But how exactly our mind perceives it, let alone manipulates it (i.e., what is the calculus of subjective probabilities?). Well, you might say : it's an old story and it has been resolved long time ago! Note that, if we adapt the current Bayesian calculus of subjective probabilities, then frequentist and Bayesian statistics are just two sides of the same coin (in the sense that they use the same calculus of probabilities).

While it has been known, also for a long time, from experimental evidence, that humans do not necessarily manipulate their subjective "probabilities" (uncertainties) according to Kolmogorov calculus, the paper [5] seems to spell it out specifically as "uncertainty appraisals" which could lead to a "radical" calculus of subjective uncertainty. We elaborate next the main message in [5], namely "Subjective probabilities are uncertainty appraisals" in social decision-making context, with emphasis on Bayesian statistics framework in econometrics.

When facing, say, an epistemic uncertainty (e.g., on an unknown parameter of a population), we take a "closer look" at it, then use any information we have about it to "appraise" it, i.e., how to describe the uncertainty a little more precise for, say, actions in a decision-making. Of course, as in general thinking processes, humans tend to use natural languages before numerical languages. As such, an appraisal of an uncertain situation should be a *linguistic variable* in Zadeh's sense [17], i.e., a variable whose possible values are words in a natural language. If the variable X is an appraisal of uncertainties, then its possible set of values could be "likely, very likely, unlikely, ..." which can be modeled as *fuzzy subsets* of the unit interval $[0, 1]$, i.e., *fuzzy probabilities.* In this quantitative modeling process, the subjective aspect of the appraisals is reflected in the shapes of membership functions of fuzzy probabilities.

When trying to model, say, the epistemic uncertainty concerning an unknown parameter θ of a population (as in standard practice of Bayesian statistics), we view θ as a linguistic variable instead. The parameter space Θ (containing the true parameter) can be coarsen into a fuzzy partition to provide granular information about θ, i.e., a set of possible linguistic values for θ, from which we could consider granular information of the form "θ is A is λ", where A is a fuzzy subset of Θ, and λ is a fuzzy probability (e.g., "θ is "small" is likely").

In order to "figure out" how to manipulate (in other words, how to derive a calculus of) uncertainty appraisals, it is necessary to dig into the modeling of membership functions of fuzzy "probabilities". Roughly speaking, saying that something is, e.g., likely, is saying that it is "possible" it is so. We used to hear statements such as "something is improbable but possible", revealing that possibility is a weaker notion than probability. Moreover, it seems that a quantitative theory of possibility was first systematically formulated by Zadeh [14]. See an update in [1]. In the next section, we will proceed to advocate that the qualitative approach to uncertainty appraisal in [5] could be quantitatively formulated by Zadeh's possibility theory.

3 POSSIBILITY MEASURES

Probability theory (Kolmogorov) is a quantitative theory of objective uncertainty. The meaning of this uncertainty is based on the notion of frequency, and its calculus (i.e., the way probabilities are combined) is based upon an analogy with measure theory in mathematics (with its main axiom of additivity). There are various different types of uncertainty, e.g., subjective probability, fuzziness. When trying to establishing a quantitative theory for a kind of uncertainty, we first examine carefully the meaning of the uncertainty under consideration, then from which, proceed to postulate its calculus (i.e., axioms). This is exemplified by Kolmogorov's formulation of (objective) probability theory.

Now, the very reason that we are concerned with uncertainty is decision-making: how we make decisions in the face of a type of uncertainty? The answer to this question dictates an appropriate calculus for that type of uncertainty. It is "interesting" to ask whether we should have an axiomatic theory of uncertainty in advance or derive it from experimental evidence?

Suppose in natural sciences, uncertainty is objective so that we could use the frequentist viewpoint to model uncertainty as (Kolmogorov) probability. Does that mean that its calculus must always obey Kolomogorov's calculus? Well, as we all know, one century ago, that was not true in quantum mechanics: while the meaning of probability is the same, the calculus of quantum probability is different than that of Kolmogorov (e.g., non additive and non commutative). Note however that quantum probability calculus is a generalization of Kolmogorov probability calculus.

Perhaps following physics, social scientists have looked at various types of uncertainties encountered in social problems since quite some time, revealing various different calculi of uncertainties (See [6] for a survey).

Now, it is well known that there is a type of uncertainty, called epistemic uncertainty in Bayesian statistics. The quantitative modeling of this uncertainty is called "subjective probability" because of the following reasons. First of all, it is subjective! The "probability" of an uncertain phenomenon is varies from person to person, and is used to quantify the uncertainty to make decisions. Secondly, it is only in your mind. And, finally, it is called "probability" since its calculus (axioms) is taken to be exactly Kolmogorov's calculus, despite their difference in semantics (subjective probabilities are not frequentist-based). We have pointed out earlier why we are led to a situation like this!

Is it the time to take a "closer look" at the calculus of subjective probabilities? say, in the "spirit" of quantum physics! Here, we examine [5] and try to figure out, in view of their "radical reformulation" of subjective probability in behavioral economics and finance, what should be a new and reasonable calculus for subjective probabilities?

Since possibility measures, as reasonable quantitative candidates for "subjective probabilities" in social sciences, seem new to social scientists, we devote this entire section to a tutorial on how the intuitive (and well-known) notion of possibility arises in scientific studies. Upfront, as stated earlier, we are going to justify Zadeh's possibility theory. Possibility theory is a mathematical theory of a weaker form of uncertainty (possibility). While the intuitive concept of possibility is familiar in everyday language, it has not been systematically used as a "scientific" tool (like probability) in, say, decision-making. Now, as mentioned above, the "acceptable" notion of "subjective probability" (as Bayesians call it) seems to resemble to possibility rather than probability (in [5], subjective probability is not probability!), it is time to make it clear: What is the calculus of subjective probability? i.e., how possibility measures are axiomatized? (just like objective probability measures are axiomatized by Kolmogorov, from which, their calculus follows). In the language of Bayesian statistics, we are basically talking about the "prior information" of the true but unknown "parameter" θ_o in the parameter space Φ. We are going to call it a "possibility distribution" of θ_o, i.e., a function $\pi(.) : \Theta \to [0, 1]$ with $\sup_{\theta \in \Theta} \pi(\theta) = 1$, where $\pi(\theta)$ represents the possibility (a value in $[0, 1]$) that θ is the true parameter. From $\pi(.)$, the notion of possibility that the true parameter is in a subset $A \subseteq \Theta$ is "postulated" to be $\pi(A) = \sup_{\theta \in A} \pi(\theta)$. This is Zadeh's theory of possibility measures [16].

We are going to justify the Zadeh's calculus of possibilities. Just like "subjective probabilities", possibilities are somewhat subjective and are used to specify "appraisals" for decision-making. They can come from your mind (in fact, based on your available information). Now, in everyday language, probability is often used to strengthen possibility, so that there is some "intuitive" link between these two notions of uncertainty. While, once the meaning of possibilities is spelled out, their calculus could be derived from the so-called "experimental evidence", we offer here a justification based on the intuitive link of probability and possibility.

Probability and possibility are somewhat related? We often say "that some phenomenon is improbable but possible", meaning that possibility is a weaker degree of belief. Typically, such statement appears in situations where we face coarse data (imprecise or low quality data). Let $(\Omega, \mathcal{A}, P)$ be a probability space. Let Θ be a set. Suppose when we perform an experiment, we observe only the outcome in a subset of Θ (rather than a precise value in it), i.e., our random experiment is described by a random set $S(.) : \Omega \to 2^{\Theta}$, so that our random variable of interest $X(.) : \Omega \to \Theta$ is an a.s. *selector* of S. An event $A \subseteq \Theta$ is realized (occured) if $X(\omega) \in A$, but if we only know that $X(\omega) \in S(\omega)$, what can we say? Well, if $S(\omega) \subseteq A$, then A occurs, but if $S(\omega) \cap A \neq \varnothing$, then it is *possible* that A occurs. Thus, we could take $\pi(A) = P(\omega : S(\omega) \cap A \neq \varnothing)$ to quantify the possibility that the event A occurs, i.e., the possibility measure of the event A. Since X is an almost sure selector of S, we have $\{\omega : X(\omega) \in A\} \subseteq \{\omega : S(\omega) \cap A \neq \varnothing\}$, and hence $P(\omega : X(\omega) \in A) \leq P(\omega : S(\omega) \cap A \neq \varnothing)$, i.e., "probability is smaller than possibility", as expected. Let $A = \{\theta\}$ be a singleton set, then the possibility of θ is $\pi(\theta) = P(\omega : S(\omega) \ni \theta)$, which is the coverage function of the random set S. Thus, formally, a possibility distribution is the coverage function of a random set.

The question now is this. What is the relation between $\pi(\theta)$ and $\pi(A)$? Of course, you are thinking about a possible analogy with probability theory in which $\pi(\theta)$ plays the role a a probability denstity function, whereas $\pi(A)$ plays the role of a probability measure.

Now, let $\pi(.) : \Theta \to [0, 1]$ such that $\sup_{\theta \in \Theta} \pi(\theta) = 1$. Then there is a random set $S(.)$ on Θ such that $P(\omega : S(\omega) \cap A \neq \varnothing) = \sup_{\theta \in A} \pi(\theta)$, for each $A \subseteq \Theta$. In other words, there is a canonical random set S whose coverage function is precisely the possibility distribution function $\pi(.)$, and the possibility measure of any event $A \subseteq \Theta$ is given as $\sup_{\theta \in A} \pi(\theta)$. Such a result justifies Zadeh's postulates $\pi(.) : \Theta \in [0, 1]$ with $\sup_{\theta \in \Theta} \pi(\theta) = 1$, and $\pi(A) =$

$\sup_{\theta \in A} \pi(\theta)$, noting that $\pi(\varnothing) = 0$.

Here is the proof. Let $\alpha(.) : (\Omega, \mathcal{A}, P) \rightarrow [0,1]$, uniformly distributed. Consider the random set $S(.) : (\Omega, \mathcal{A}, P) \rightarrow 2^{\Theta}$ which is the randomized level set of the function $\pi(.) : \Theta \rightarrow [0,1]$, i.e.,

$$S(\omega) = \{\theta \in \Theta : \pi(\theta) \geq \alpha(\omega)\},$$

Then, for $A \subseteq \Theta$, we have

$$\begin{aligned}\pi(A) &= P(\omega : S(\omega) \cap A \neq \varnothing) \\ &= P\{\omega : \alpha(\omega) \leq \sup_{\theta \in A} \pi(\theta)\} \\ &= \sup_{\theta \in A} \pi(\theta),\end{aligned}$$

Remark. From $\pi(A) = \sup_{\theta \in A} \pi(\theta)$, we see that, in particular, the possibility measure $\pi(.)$ on 2^{Θ}, is *maxitive* (rather than additive), i.e., for any A, B, we have $\pi(A \cup B) = \max\{\pi(A), \pi(B)\}$.It is interesting to note that maxitive operator can arise from additive operator in some limiting process. Let $P_n, n \geq 1$ be a sequence of probability measures on a measurable space $(\Omega, \mathcal{A})$, and $\varepsilon_n > 0$, with $\varepsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Then it is possible that the sequence of submeasures $P_n^{\varepsilon_n}, n \geq 1$ converges to a $(\sigma-)$ maxitive set-function. Here is an example. Let $(\Omega, \mathcal{A}, P)$ be a probability space, and $f(.) : \Omega \rightarrow \mathbb{R}^{+}$, $f \in L^{\infty}(\Omega, \mathcal{A}, P)$. Since P is a finite measure, it follows that $f \in L^{p}(\Omega, \mathcal{A}, P)$, for all $p > 0$. Consider

$$P_n(A) = \frac{\int_A |f(\omega)|^n dP}{\int_\Omega |f(\omega)|^n dP},$$

then, as $n \rightarrow \infty$,

$$[P_n(A)]^{\frac{1}{n}} = \frac{||f 1_A||_n}{||f||_n} \rightarrow \frac{||f 1_A||_\infty}{||f||_\infty},$$

where the set function $\pi(A) = \dfrac{||f 1_A||_\infty}{||f||_\infty}$ is $(\sigma-)$ maxitive.

Of course, if we "consider" a function $\pi(.) : \Theta \rightarrow [0,1]$ with $\sup_{\theta \in \Theta} \pi(\theta) = 1$, and call it our possibility distribution (just like "probability density", where "possibility" reflects our "subjective" assignments/ appraisals), then our possibility measure of $A \subseteq \Theta$ is taken to be $\pi(A) = \sup_{\theta \in A} \pi(\theta)$, although we do not need to relate it to the notion of probability. In other words, uncertainties might not need to be a function of probability. For example, a function $\pi(.)$ as above plays the role of a membership function of a fuzzy subset of Θ, generalizing ordinary subsets of Θ (see [15] for more discussions on how to obtain possibility distributions in applications). What we have done above is to give a meaning to the concept of possibility distribution (from which possibility measures are built), namely "a possibility distribution is the coverage function of a random set". If you refer to the concept of confidence intervals in statistics, then you realize the flavor of "subjective" assessments.

It is interesting to note that, formally, the concept of possibility distribution is somewhat "hidden" within probability theory! In fact, the notion of possibility distribution "appears" also in probability theory as limits of probabilities. When X is an absolutely continuous random variable, e.g., taking values, say, in $\mathbb{R} = (-\infty, \infty)$, with distribution function $F(x) = P\{\omega : X(\omega) \leq x\}$, then its density function value

$$f(x) = \lim_{h \to 0} \frac{F(x+h) - F(x-h)}{2h}$$
$$= \lim_{h \to 0} \frac{P\{\omega : x - h \leq X(\omega) \leq x + h\})}{2h}$$

can be interpreted as a possibility value (and not a probability value). For example, in the context of Bayesian statistics, when the parameter space is $\Theta = [0, 1]$, the uniform prior density $f(\theta) = 1$, for any $\theta \in [0, 1]$ is a possibility distribution, resulting in $\sup_{\theta \in A} f(\theta) = 1$ for any $A \subseteq [0, 1]$. And this can be applied to unbounded parameter spaces like $\mathbb{R} = (-\infty, \infty)$ with a possibility distribution $f(\theta) = 1$, for any $\theta \in \mathbb{R}$, without "calling" it an improper prior probability density function !

To be complete, we mention here that possibility measures can be also interpreted as limits of probability measures in the large deviation convergence sense (See [8], [11]).

Recall that the purpose of the study of large deviations in probability theory is to find the asymptotic rate of convergence of sequences of probability measures of rare events. For example, in the simplest setting, while the law of large numbers asserts that the sequence of sample means (of i.i.d. $X_j, j \geq 1$, with $EX = \mu, Var(X) = \sigma^2 < \infty$) $\bar{X}_n$ converges surely to μ, it is also of interest to find the rate at which $P(|\bar{X}_n - \mu| > a)$ (probability of large deviations from the mean) goes to zero, as $n \to \infty$, for some a.

Now, for $Z_n = \sqrt{n}(\bar{X}_n - \mu)/\sigma$, we have, by the Central Limit Theorem,

$$P(Z_n > x) \approx \int_x^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} dt$$

By finding the asymptotics of this integral for $x \to \infty$, we arrive at the well-known Cramer result of the form : The sequence of sample means satisfies the large deviation principle in the sense that, for $\varepsilon > 0$, we have

$$P(|\bar{X}_n - \mu| > a) \leq e^{-nh(a)+\varepsilon}$$

for n sufficiently large, where the exponential rate of convergence being $e^{-nh(a)}$.

The general setting of large deviation principle is this. Let Θ be a complete, separable metric space, and $\mathcal{B}$ its borel $\sigma-$field. A sequence of probability measures P_n on $(\Theta, \mathcal{B})$ is said to obey the large deviation principle (LDP) if there exists a lower semicontinuous function $I(.) : \Theta \to [0, \infty]$ (the rate function) such that :

(i) for each closed set F of Θ,

$$\limsup_{n \to \infty} [\frac{1}{n} \log P_n(F)] \leq - \inf_{\theta \in F} I(\theta)$$

(ii) for each open set G of Θ,

$$\limsup_{n \to \infty} [\frac{1}{n} \log P_n(G)] \geq - \inf_{\theta \in G} I(\theta)$$

If we let $\varphi(A) = \sup_{\theta \in A} e^{-I(\theta)}$, then the LDP is formulated as

$$\limsup_{n \to \infty} [P_n(F)]^{\frac{1}{n}} \leq \varphi(F)$$

and

$$\limsup_{n \to \infty} [P_n(G)]^{\frac{1}{n}} \leq \varphi(G)$$

If we refer the above to the Portmamteau theorem of weak convergence of probability measures, we can view the set function $\varphi(.)$ as a limit of P_n *in the sense of LDP.*

Remark. Since the possibility distribution $\varphi(\theta) = e^{-I(\theta)}$ is upper semicontinuous (with values in $[0,1]$), the associated possibility measure $\varphi(.)$ characterizes the probability distribution of a *random closed sets on* Θ (See [9] for details).

In summary, a possibility distribution (for an appraisal) is a function $\pi(.)$ on an arbitrary set Θ, taking values in the unit interval $[0,1]$, such that $\sup_{\theta\in\Theta}\pi(\theta) = 1$. Its associated possibility measure, denoted also as $\pi(.)$, is a map from the power set 2^Θ of Θ to $[0,1]$, defined by $\pi(A) = \sup_{\theta\in A}\pi(\theta)$, for $A \subseteq \Theta$. Equivalently, a possibility measure is a map $\pi(.) : 2^\Theta \to [0,1]$ such that $\pi(\varnothing) = 0$, $\pi(\Theta) = 1$, and $\pi(\cup_{i\in I} A_i) = \sup_{i\in I}\pi(A_i)$, for any index set I.

Remark. For a general theory of idempotent probability, see [11].

4 BAYESIAN ANALYSIS WITH POSSIBILITY MEASURES

First of all, in the context of Bayesian statistics, the additional information required is a prior probability measure (or often a probability density function) on the parameter space, reflecting the subjective information about the possible location of the true (but unknown) parameter of the population. Such an information is quite compatible with a possibility distribution where the value of the possibility distribution of an element in the parameter space acts as the degree of membership of that element in the statistician's perception.

Secondly, while the literature did contain research works on the possibility to consider "Bayesian statistics" where both prior and objective probability measures are replaced by possibility measures, it seems that the more practical situation where the prior subjective (epistemic) uncertainty is possibilistic, but the objective uncertainty (in the statistical models) remains Kolmogorov is not completely worked out. This is the situation where two different types of uncertainty calculi need to be combined!

The paper [5] is, in a sense, a confirmation of [2], from an original idea in [16]. And now, with all necessary justifications spelled ont in the previous section, it is about time to see how "traditional" Bayesian statistics is affected?! This will happen clearly when we simply replace the calculus of subjective probabilities by the calculus of possibilities in the formulation of the "Bayes theorem", combining objective probability with subjective probability.

A situation closely related to what we wish to discuss here is robust Bayesian statistics, as exemplified by [12]. In a sense, robust Bayesian inference refers to the situation where we consider a set of possible priors (say, as prior probability measures on the parameter space) rather just one given one. Specifically, let $\mathcal{P}$ be a set of probability measures on $(\Theta, \mathcal{B}(\Theta))$. Without knowing a specific prior in $\mathcal{P}$, we must work with the lower and upper envelops of $\mathcal{P}$, i.e., the non-additive set-functions $L(.) = \inf_{P\in\mathcal{P}} P(.)$, $U(.) = \sup_{P\in\mathcal{P}} P(.)$, respectively. However, for each $P \in \mathcal{P}$, we can use Bayes theorem to get the corresponding posterior probability measure. Bounds on these posterior probability measures can be obtained from $L(.)$ and $U(.)$. Specifically, as in [12], these bounds are expressed as (Choquet) integrals of monotone increasing set functions (called Choquet capacities) $L(.)$ and $U(.)$, as a generalization of ordinary integral calculus.

Now, in one hand, possibility measures are Choquet capacities, and on the other hand, if we view a possibility measure $\pi(.)$ as the envelop of a set of prior probability measures $\mathcal{P}$, i.e., say, $\mathcal{P} = \{P : P(.) \leq \pi(.)\}$, then we can proceed as in [12], noting that a possibility measure $\pi(.)$ is , in particular, maxitive, i.e., for A, B in 2^{Θ}, $\pi(A\cup B) = \max\{\pi(A), \pi(B)\}$. But, any such set-function is "alternating of infinite order" (and hence, in particular, 2-alternating), i.e., for any $A_1, A_2, ..., A_n$, we have

$$\pi(\cap_{i=1}^{n} A_i) \leq \sum_{\varnothing\neq I\subseteq\{1,2,\ldots,n\}} (-1)^{|I|+1}\pi(\cup_{i\in I} A_i)$$

where $|I|$ denotes the cardinality of the set I. For a proof, see [9].

In general, the problem seems open.

References

[1] Dubois, D., & Prade, H. (2011). Possibility theory and its applications: Where do we stand? *Mathware and soft computing*, 18(1), 18–31.

[2] Dubois, D. (2006). Possibility theory and statistical reasoning. *Computational Statistics and Data Analysis*, 51, 47-69. DOI:10.1016/j.csda.2006.04.015.

[3] Goodman, I. R., & Nguyen, H. T. (1985). *Uncertainty Models for Knowledge-Based Systems,* North-Holland

[4] Goodman, I. R. , Nguyen, H. T., & Rogers, G. S. (1991). On the scoring approach to admissibility of uncertainty measures in expert systems, *J. Math. Anal. and Appl.*, 159, 550–594.

[5] Jeffrey, H. J., & Putman, A. O. (2015). Subjective probability on behavioral economics and finance: A radical reformulation, *J. Behavioral Finance*, 16, 231–249.

[6] Kreps, D. V. (1988). *Notes on The Theory of Choice.* Westview Press.

[7] Lindley, D. V. (1982). Scoring rules and the inevitability of probability. *Intern. Statist. Review*, 50(1), 1–11.

[8] Nguyen, H. T., & Bouchon-Meunier, B. (2003). Random sets and large deviations principle as a foundation for possibility measures. *Soft Computing*, 8, 61–70.

[9] Nguyen, H. T. (2006). *An Introduction to Random Sets.* Chapman and Hall / CRC Press.

[10] Nguyen, H. T., Walker, L. C., & Walker, E. A. (2019). *A First Course in Fuzzy Logic* (Fourth Edition). Chapman and Hall / CRC Press.

[11] Puhalskii, A. (2001) *Large Deviations and Idempotent Probability.* Chapman and Hall / CRC Press.

[12] Wasserman, L. A., & Kadane, J. B. (1990). Bayes' theorem for Choquet capacities. *Ann. Statist.*, 18(3), 1328–1339.

[13] Yao, J. T., Vasilakos, A. V., & Pedrycz, W. (2013). Granular computing: Perspectives and challenges. *IEEE Trans Cybern.*, 34(6), 1977–1989.

[14] Yao, M. X. (2019). Granularity measures and complexity of partition-based granular structures. *Knowledge-Based Systems*, 163, 885–897.

[15] Zadeh, L. A. (1996). Fuzzy sets and information granularity. In G.J. Klir and B. Yuan (Eds), *Fuzzy Sets, Fuzzy Logic and Fuzzy Systems (Selected papers of Lotfi A. Zadeh)* (433-448). World Scientific.

[16] Zadeh, L. A. (1978). Fuzzy sets as a basis for a theory of possibility. *J. Fuzzy Sets and Systems*, 1, 3–28.

[17] Zadeh, L.A. (1975). The concept of a linguistic variable and its application to approximate reasoning. *I (Information Sci., 8, 199-249), II (Infomation Sci. 8, 301-357), III (Information Sci. 9, 43-80)*