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Nonlinear elasticity with geometric linearity

Độ đàn hồi phi tuyến với sự tuyến tính hình học

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Abstract

We study some properties involving the strain energy for a nonlinear elasticity problem with geometric linearity in one-dimensional and strain-limiting settings.

Keywords: Strain energy; nonlinear elasticity; geometric linearity; strain-limiting.

Tóm tắt

Chúng tôi nghiên cứu một vài đặc tính của năng lượng biến dạng cho một bài toán độ đàn hồi phi tuyến với sự tuyến tính hình học trong thiết lập một chiều và giới hạn biến dạng.

Từ khóa: Năng lượng biến dạng; độ đàn hồi phi tuyến; sự tuyến tính hình học; giới hạn biến dạng.

1. Introduction

We consider in this paper a displacement problem in strain-limiting theory of nonlinear elasticity as introduced in [1, 2]. In particular, we study the properties of strain energy for a nonlinear elasticity problem with geometric linearity in one-dimensional and strain-limiting settings.

2. Formulation of the problem

2.1. Classical formulation

We consider herein a spatially 1D composite rod formed by nonlinear elastic material, which is computationally denoted by Ω. Assume that Ω is a bounded, connected, open, Lipschitz domain of $\mathbb{R}$. The boundary of the set Ω is represented by $\partial\Omega$, which is Lipschitz continuous, consisting of two parts $\partial\Omega_T$ and $\partial\Omega_D$.

For simplicity, the rod is assumed to be at a static state after the action of body forces (along the rod) $f : \Omega \to \mathbb{R}$ and traction forces $G : \partial\Omega_T \to \mathbb{R}$. The displacement $u : \Omega \to \mathbb{R}$ is considered on $\partial\Omega_D$. We are investigating the strain-limiting model of the following form (as in [1]):

$$E = \frac{\sigma}{1 + \beta|\sigma|}. \tag{1}$$

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That is,

$$\sigma = \frac{E}{1-\beta|E|}. \tag{2}$$

In Eqs. (1) and (2), β is the strain-limiting parameter (which will be discussed in the next paragraph), σ stands for the Cauchy stress $\sigma : \Omega \to \mathbb{R}$, and E denotes the classical linearized strain tensor (which implies geometric linearity)

$$E := \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}}). \tag{3}$$

In one-dimensional setting,

$$E := u', \tag{4}$$

saying, the spatial derivative of u. Therefore, by (2), we obtain

$$\sigma = \frac{u'}{1-\beta|u'|}. \tag{5}$$

We derive from (1) that

$$|E| = \frac{|\sigma|}{1+\beta|\sigma|} < \frac{1}{\beta}. \tag{6}$$

This shows that $\frac{1}{\beta}$ is the upper-bound on $|E|$ and choosing sufficiently large β produces small upper-bound on the limiting-strain, as desired. Nevertheless, we refrain from too large β. If $\beta \to \infty$ then $|E| < \frac{1}{\beta} \to 0$, which is not an expected behavior. In this paper, β is taken so that the strong ellipticity condition [1] is attained, to prevent bifurcations arising in numerical simulations.

2.2. *Function spaces*

Let $V := H_0^1(\Omega)$ is our needed space. Nevertheless, the methods in this paper can be extended to more general space $H_0^p(\Omega)$, where $2 \le p < \infty$. The space $W_0^{1,2}(\Omega)$ is of interests because it can help handle displacements that vanish on the boundary $\partial\Omega$ of Ω.

Let $H^{-1}(\Omega)$ be the dual space, which is the space of continuous linear functionals on $H_0^1(\Omega)$, and the value of a functional $b \in H^{-1}(\Omega)$ at a point $v \in H_0^1(\Omega)$ is denoted by $\langle b, v\rangle$. The Sobolev norm $\|\cdot\|_{H_0^1(\Omega)}$ is of the following form:

$$\|v\|_{H_0^1(\Omega)} = \left(\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}}.$$

The dual norm to $\|\cdot\|_{H_0^1(\Omega)}$ is represented by $\|\cdot\|_{H^{-1}(\Omega)}$.

We define

$$f \in H_*^1(\Omega) = \left\{ g \in H^1(\Omega) \,\middle|\, \int_\Omega g\, dx = 0 \right\}.$$

The following problem is of our interest: Find $u \in H^1(\Omega)$ and $\sigma \in L^1(\Omega)$ ([3]) such that

$$\begin{aligned} -\mathrm{div}(\sigma) &= f \quad \text{in } \Omega, \\ \sigma &= \frac{u'}{1-\beta|u'|} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega_D, \\ \sigma &= G \quad \text{on } \partial\Omega_T. \end{aligned} \tag{7}$$

Here, we assume that $\partial\Omega_T = \emptyset$. Using (7), we rewrite the considered formulation in the form of displacement problem: Find $u \in H^1(\Omega)$ such that

$$-\mathrm{div}\left(\frac{u'}{1-\beta|u'|}\right) = f \quad \text{in } \Omega, \tag{8}$$

$$u = 0 \quad \text{on } \partial\Omega. \tag{9}$$

Let

$$a(x, u') = \frac{u'}{1-\beta|u'|}, \tag{10}$$

in which $u(x) \in W_0^{1,2}(\Omega)$.

3. Existence and uniqueness

In [4], the existence and uniqueness of solution to (8)-(9) is proved and thanks to the following Lemma ([5, 6, 4]).

Lemma 3.1. *Let*

$$\mathcal{Z} := \left\{ \zeta \in L^\infty(\Omega) \,\middle|\, 0 \le |\zeta| < \frac{1}{\beta} \right\}. \tag{11}$$

For any $\xi \in \mathcal{Z}$, we consider the mapping

$$\xi \in \mathcal{Z} \mapsto F(\xi) := \frac{\xi}{1-\beta|\xi|} \in \mathbb{R}.$$

Then, for each $\xi_1, \xi_2 \in \mathcal{Z}$, we get

$$|F(\xi_1) - F(\xi_2)| \le \frac{|\xi_1 - \xi_2|}{(1-\beta(|\xi_1|+|\xi_2|))^2}, \tag{12}$$

$$(F(\xi_1) - F(\xi_2))(\xi_1 - \xi_2) \ge |\xi_1 - \xi_2|^2. \tag{13}$$

In our case of 1D, the solution u can be found directly from (8)-(9).

4. Hyperelasticity

The considered model (1) is compatible with the laws of thermodynamics [7, 8], which means that the class of materials are non-dissipative and elastic. Moreover, this class of materials is hyperelastic [1, 9]. Specifically, the relation (2) can be derived from the strain energy function

$$\hat{h}(E) = \tilde{h}(|E|)\,, \tag{14}$$

with

$$\tilde{h}(r) := \int \frac{r}{1-\beta r}\, dr\,. \tag{15}$$

It is simple to verify that

$$\sigma = \partial_E \hat{h}(E) = \partial_E \tilde{h}(|E|)\,.$$

In our case, the strain energy function obtained in [1, 9] is of the form

$$\tilde{h}(r) := -\frac{1}{\beta^2}(\ln(1-\beta r) + \beta r)\,. \tag{16}$$

The complementary energy function is defined through Legendre transformation of the strain energy:

$$\hat{k}(\sigma) := -\hat{h}(E) + \sigma \cdot E = \tilde{k}(|\sigma|)\,. \tag{17}$$

In our setting, (17) has the form

$$\tilde{k}(r) := \frac{1}{\beta^2}(\beta r - \ln(1+\beta r))\,. \tag{18}$$

5. Some properties of strain energy

Thanks to [9], we consider the expression

$$J(v) = \int_\Omega \tilde{h}(|Dv|)\, dx - L(v)$$
$$= \int_\Omega \left(-\frac{1}{\beta^2}\right)[\ln(1-\beta|Dv|) + \beta|Dv|]\, dx - L(v)\,, \tag{19}$$

where

$$L(v) = \int_\Omega f\, v\, dx\,,$$

$$V := H_0^1(\Omega) = \left\{v \in H^1(\Omega)\,\middle|\, \mathrm{tr} v = 0 \text{ on } \partial\Omega\right\}.$$

The following properties of $J(v)$ readily hold.

Lemma 5.1. *$J(v)$ is proper, strictly convex, and continuous on V.*

Proof. We prove, for instance, the strict convexity. Regarding the last two summands in (19), the convexity of $\beta|Dv|$ and $L(v)$ comes from their linearity on V.

For the first summand of (19), the strict convexity follows from the fact that the increasing and strictly convex function $k_1(y) = \left(-\frac{1}{\beta^2}\right)\ln(1-\beta y)$ $\left(\forall\, 0 \le y < \frac{1}{\beta}\right)$ combining with the convex function $k_2(v) = |Dv|$ produces a strictly convex function. More specifically, the first and second derivatives of $k_1(y)$ with respect to y are both positive:

$$(k_1(y))' = \left(-\frac{1}{\beta^2}\ln(1-\beta y)\right)' = \frac{1}{\beta(1-\beta y)} > 0$$

(increasing of k_1),

$$(k_1(y))'' = \left(\frac{1}{\beta(1-\beta y)}\right)' = \frac{1}{(1-\beta y)^2} > 0$$

(strict convexity of k_1).

Now, we want to show that for all $t \in [0,1]$ and $v, w \in V$,

$$(k_1 \circ k_2)(tv + (1-t)w)$$
$$< t(k_1 \circ k_2)(v) + (1-t)(k_1 \circ k_2)(w)\,.$$

It is clear that

$$\begin{aligned}
&(k_1 \circ k_2)(tv + (1-t)w)\\
&= k_1(k_2(tv + (1-t)w))\\
&\le k_1(tk_2(v) + (1-t)k_2(w))\\
&< tk_1(k_2(v)) + (1-t)k_1(k_2(w))\\
&= t(k_1 \circ k_2)(v) + (1-t)(k_1 \circ k_2)(w)\,,
\end{aligned}$$

and we are done.

Remark 5.2. *As a consequence of Lemma 5.1, for the minimization problem*

$$J(u) = \inf_{v \in V} J(v)\,,$$

it is well-known (see [10, 11], for instance) that the unknown displacement vector field $u : \Omega \to \mathbb{R}^3$ is the unique solution.

With the given displacement problem in three-dimensional strain-limiting theory of elasticity, this minimization problem, instead of being called *principle of minimum potential energy*, will be modernly referred to as the *displacement formulation* [12].

6. Conclusions

In this paper, we investigate the properties of strain energy for a nonlinear elasticity problem with geometric linearity in one-dimensional and strain-limiting settings. The results here still hold in higher dimensions (for example, three). An open question is extending this study to the (complementary) stress energy.

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