TẠP CHÍ KHOA HỌC & CÔNG NGHỆ ĐẠI HỌC DUY TÂN
DTU Journal of Science and Technology 07(38) (2020) ....

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Analysis of multiscale finite element method for a periodically nonlinear elasticity problem

Phương pháp phần tử hữu hạn đa kích thước cho bài toán đàn hồi phi tuyến tuần hoàn

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(Ngày nhận bài: 18/10/2019, ngày phản biện xong: 12/11/2019, ngày chấp nhận đăng: 20/02/2020)

Abstract

We study analysis (including convergence) of the multiscale finite element method (MsFEM) for a periodically nonlinear elasticity problem in one-dimensional and strain-limiting settings.

Keywords: Analysis, convergence, multiscale finite element method, homogenization, periodic, nonlinear elasticity, strain-limiting.

Tóm tắt

Chúng tôi nghiên cứu giải tích (bao gồm sự hội tụ) của phương pháp phần tử hữu hạn đa kích thước (MsFEM) cho một bài toán độ đàn hồi phi tuyến tuần hoàn trong thiết lập một chiều và giới hạn biến dạng.

Từ khóa: Giải tích, sự hội tụ, phương pháp phần tử hữu hạn đa kích thước, đồng nhất hóa, tuần hoàn, độ đàn hồi phi tuyến, giới hạn biến dạng.

1. Introduction

As a model reduction approach for tackling multiscale problems, the multiscale finite element method (MsFEM) involves finding a numerical approximation of a homogenized solution *without* solving auxiliary problems (for example, periodic cell problems) that arise in homogenization. Toward such multiscale investigation of our considering nonlinear elasticity models, we focus on a periodically strain-limiting problem. (The strain-limiting parameter in this paper is a function depending on the position variable, which is different from the constant in [1, 2].) In particular, we study the analysis (including convergence) of the MsFEM for a periodically nonlinear elasticity problem in one-dimensional and strain-limiting settings.

2. Formulation of the problem

2.1. Classical formulation

As in Figure 1, we consider in the x-direction, the spatially periodic 1D composite

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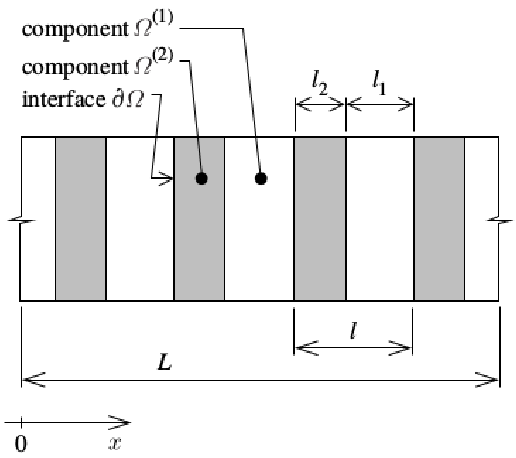

Hình 1. Layered composite structure (from [3]).

rod consisting of alternating layers of nonlinear elastic media $\Omega^{(1)}$ and $\Omega^{(2)}$. The microscopic size corresponds to the length of a periodically repeated base cell is represented by l. The macroscopic size of the total sampling $\Omega \subset \mathbb{R}$ of the rod is denoted by L. Without loss of generality, we pick $l = \epsilon$ (the period of the structure) and choose $L = 1$ so that

$$\epsilon = \frac{l}{L} = \frac{\epsilon}{1} = \frac{k\epsilon}{k1}. \tag{1}$$

Here, $\frac{x}{\epsilon}$ denotes the local position.

The rod is assumed to be at a static state after the action of body forces (along the rod) $f : \Omega \to \mathbb{R}$ and traction forces $G : \partial\Omega_T \to \mathbb{R}$. The boundary of the set Ω is denoted by $\partial\Omega$, which is Lipschitz continuous, having two parts $\partial\Omega_T$ and $\partial\Omega_D$, where the displacement $u : \Omega \to \mathbb{R}$ is provided on $\partial\Omega_D$. We are focusing on the strain-limiting model of the form (as in [1])

$$E = \frac{\sigma}{1 + \beta(x)|\sigma|}. \tag{2}$$

Equivalently,

$$\sigma = \frac{E}{1 - \beta(x)|E|}. \tag{3}$$

In Eqs. (2) and (3), $\beta(x)$ will be described in the next paragraph, σ denotes the Cauchy stress $\sigma : \Omega \to \mathbb{R}$; and E stands for the classical linearized strain tensor

$$E := \frac{1}{2}(\nabla u + \nabla u^{\mathrm{T}}). \tag{4}$$

In one-dimensional setting, it is

$$E := u', \tag{5}$$

namely, the spatial derivative of u. Hence, by (3),

$$\sigma = \frac{u'}{1 - \beta(x)|u'|}. \tag{6}$$

The strain-limiting parameter function is represented by $\beta(x)$, which depends on the position variable x, and it is constant with respect to each layer, with $\beta_\epsilon(x) = \beta(\epsilon^{-1}x)$. We obtain from (2) that

$$|E| = \frac{|\sigma|}{1 + \beta(x)|\sigma|} < \frac{1}{\beta(x)}. \tag{7}$$

This indicates that $\frac{1}{\beta(x)}$ is the upper-bound on $|E|$ and taking sufficiently big $\beta(x)$ brings the limiting-strain small upper-bound, as expected. However, too large $\beta(x)$ is avoided. If $\beta(x) \to \infty$ then $|E| < \frac{1}{\beta(x)} \to 0$, a contradiction. Moreover, $\beta(x)$ is assumed to be smooth and have compact range $0 < m \le \beta(x) \le M$. Also, we assume that

$$\beta(x) = \begin{cases} \beta_1 & \text{if } jl < x < (j + \alpha)l \text{ for some } j \in \mathbb{N}, \\ \beta_2 & \text{otherwise}. \end{cases} \tag{8}$$

Here, β_1 and β_2 are selected so that the strong ellipticity condition [1] is met. Realistically, the requirement of strong point-wise ellipticity in each layer is not vital. This happens because all the crucial instability phenomena occur somewhat below the stress levels corresponding to the loss of ellipticity of the weakest layer (see [4, 5]).

2.2. Function spaces

Let $V := H_0^1(\Omega)$ is our considered space. Even so, the approaches here can be applied to more general space $H_0^p(\Omega)$, where $2 \le p < \infty$. The space $W_0^{1,2}(\Omega)$ is of attention because we can capture displacements that vanish on the boundary $\partial\Omega$ of Ω.

Let $H^{-1}(\Omega)$ be the dual space, which is the space of continuous linear functionals on $H_0^1(\Omega)$, and the value of a functional $b \in H^{-1}(\Omega)$ at a point $v \in H_0^1(\Omega)$ is represented by $\langle b, v\rangle$. The Sobolev norm $\|\cdot\|_{H_0^1(\Omega)}$ is of the form

$$\|v\|_{H_0^1(\Omega)} = \left(\|v\|^2_{L^2(\Omega)} + \|\nabla v\|^2_{L^2(\Omega)}\right)^{\frac{1}{2}} .$$

The dual norm to $\|\cdot\|_{H_0^1(\Omega)}$ is $\|\cdot\|_{H^{-1}(\Omega)}$.

Let Ω be a bounded, connected, open, Lipschitz domain of $\mathbb{R}$,

$$f \in H_*^1(\Omega) = \left\{ g \in H^1(\Omega) \,\middle|\, \int_\Omega g\, dx = 0 \right\}.$$

The following problem is of our consideration: Find $u \in H^1(\Omega)$ and $\sigma \in L^1(\Omega)$ ([6]) such that

$$\begin{aligned} -\mathrm{div}(\sigma) &= f \quad \text{in } \Omega, \\ \sigma &= \frac{u'}{1-\beta(x)|u'|} \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega_D, \\ \sigma &= G \quad \text{on } \partial\Omega_T. \end{aligned} \tag{9}$$

The considered model (2) is compatible with the laws of thermodynamics [7, 8], that is, the class of materials are elastic and non-dissipative.

For the later use, we consider $u_\epsilon(x) \in W_0^{1,2}(\Omega)$. Assume that $u_\epsilon(x) = u\left(\frac{x}{\epsilon}\right)$ is a periodic function in x with period ϵ. Equivalently, $u(y) = u\left(\frac{x}{\epsilon}\right)$ is a periodic function in y with period 1. This implies that for any integer k,

$$u_\epsilon(x) = u_\epsilon(x+\epsilon) = u_\epsilon(x+k\epsilon),$$

correspondingly,

$$u\left(\frac{x}{\epsilon}\right) = u\left(\frac{x}{\epsilon}+1\right) = u\left(\frac{x}{\epsilon}+k1\right) = u(y+k).$$

This observation assists the expressions of ϵ in (1). (Note that the spatial periodicity of the composite leads to the same periodicity for u.)

For advantage, we assume perfect bonding conditions at the interface $\partial\Omega$ between the layers, that is, the displacement and traction are continuous across each interface for all feasible deformations:

$$\begin{aligned} (u_\epsilon)_{(1)} &= (u_\epsilon)_{(2)} \quad \text{on } \partial\Omega, \\ (\sigma_\epsilon)_{(1)} &= (\sigma_\epsilon)_{(2)} \quad \text{on } \partial\Omega_T. \end{aligned} \tag{10}$$

Assume that $\partial\Omega_T = \emptyset$. In homogenization theory, using (9), we rewrite the considered formulation in the form of displacement problem: Find $u \in H^1(\Omega)$ such that

$$-\mathrm{div}\left(\frac{u'_\epsilon}{1-\beta_\epsilon(x)\left|u'_\epsilon\right|}\right) = f \quad \text{in } \Omega, \tag{11}$$

$$u_\epsilon = 0 \quad (u_\epsilon)_{(1)} = (u_\epsilon)_{(2)} \quad \text{on } \partial\Omega. \tag{12}$$

Let

$$a_\epsilon(x, u'_\epsilon) = \frac{u'_\epsilon}{1-\beta_\epsilon(x)\left|u'_\epsilon\right|}, \tag{13}$$

in which $u_\epsilon(x) \in W_0^{1,2}(\Omega)$.

3. Existence and uniqueness

In [9], the existence and uniqueness of solution to (11)-(12) is proved and thanks to the following Lemma ([10, 11, 9]).

Lemma 3.1. *Let*

$$\mathcal{Z} := \left\{ \zeta \in L^\infty(\Omega) \,\middle|\, 0 \le |\zeta| < \frac{1}{M} \right\}. \tag{14}$$

For any $\xi \in \mathcal{Z}$, we consider the mapping

$$\xi \in \mathcal{Z} \mapsto F(\xi) := \frac{\xi}{1-\beta_\epsilon(x)|\xi|} \in \mathbb{R}.$$

Then, for each $\xi_1, \xi_2 \in \mathcal{Z}$, we get

$$|F(\xi_1) - F(\xi_2)| \le \frac{|\xi_1 - \xi_2|}{(1-\beta_\epsilon(x)(|\xi_1|+|\xi_2|))^2}, \tag{15}$$

$$(F(\xi_1) - F(\xi_2))(\xi_1 - \xi_2) \ge |\xi_1 - \xi_2|^2. \tag{16}$$

In our case of 1D, the solution u can be solved directly from (11)-(12).

4. Multiscale finite element method (MsFEM)

The goal of the MsFEM [12, 13] is to find a numerical approximation of a homogenized solution *without* solving auxiliary problems (for example, periodic cell problems) that arise in homogenization. In our setting, we consider formulation and analysis of the MsFEM for (11)-(13), in which $u_\epsilon(x) \in W_0^{1,2}(\Omega)$. Here, $a_\epsilon\left(x, u'_\epsilon\right)$ as in (13) satisfies the assumptions (15) and (16). We use ϵ to denote the fine-scale problem and fine-scale quantities.

Basis functions. Without loss of generality, let K^h be a usual partition of $\Omega = [0, 1]$ into finite elements (segments or blocks).

$$0 = x_0 < x_1 < \cdots < x_i < x_{i+1} < \cdots < x_N = 1 . \tag{17}$$

Each segment K_i of the form $[x_{i-1}, x_i] (i = 1, \cdots N)$ in K^h is of length $1/N$. We call this partition the *coarse grid* and assume that the coarse grid can be resolved via a finer resolution called the *fine grid*. Each coarse-grid block is a connected union of fine-grid blocks. Let x_i be the interior nodes of the mesh K^h and ϕ_i^0 be the nodal basis of the standard finite element space

$$W_h = \mathrm{span}\left\{\phi_i^0 : i = 1, \cdots N; K_i \in K^h\right\} \subset W_0^{1,2}(\Omega) . \tag{18}$$

Note that even though the choice of ϕ_i^0 can be quite arbitrary, our main assumption is that ϕ_i^0 satisfies

$$\left(a_\epsilon\left(x, (\phi_i^0)'\right)\right)' = 0 \quad \text{in } K_i \in K^h , \tag{19}$$

that is

$$\int_\Omega a_\epsilon\left(x, (\phi_i^0)'\right) v'(x)\, dx = 0 \ \forall\, v \in H_0^1(K_i \cup K_{i+1}) . \tag{20}$$

As usual, we require $\phi_i^0(x_j) = \delta_{ij}$.

Note also that ϕ_i^0 is clearly continuous on Ω. Moreover, although ϕ_i^0 has discontinuities in slope across element boundaries, it is smooth enough that $\phi_i^0 \in H_0^1(\Omega)$, so it leads to a conforming approximation space (18).

For simplicity, ones can assume that W_h consists of piecewise linear functions (although this is not usually the case for the true solution $u(x)$):

$$W_h = \Big\{ v_h \in C^0(\bar\Omega) : \text{ the restriction } v_h \text{ is linear for each segment } K_i \in K^h, v_h = 0 \text{ on } \partial\Omega \Big\} . \tag{21}$$

Multiscale mapping. Unlike the MsFEM for linear problems, "basis" functions for nonlinear problems need to be defined via nonlinear maps that map coarse-scale functions into fine-scale functions. We introduce the mapping $E^{MsFEM} : W_h \to V_\epsilon^h$ in the following way. For each $v_h \in W_h$, $v_{\epsilon,h} = E^{MsFEM} v_h$ is defined as the solution of the local problem

$$\left(a_\epsilon\left(x, v'_{\epsilon,h}\right)\right)' = 0 \quad \text{in } K_i \in K^h , \tag{22}$$

where $v_{\epsilon,h} = v_h$ on ∂K_i, that is, for every interior node x_i of Ω. In each block $K_i = [x_{i-1}, x_i]$, (22) can be solved, in fine grid, in general. Throughout the paragraphs below, we can obtain an explicit expression for E^{MsFEM}. The map E^{MsFEM} is nonlinear, but, for a fixed v_h on $K_i \in K^h$, this map is linear. In fact, one can represent $v_{\epsilon,h}$ using multiscale basis functions as $v_{\epsilon,h} = \sum_{i=1}^{N} v_i \phi_i^{v_h}$, where $v_i = v_h(x_i)$, x_i being nodal points, and $\phi_i^{v_h}$ are multiscale basis functions defined by

$$\begin{aligned} \left(a_\epsilon\left(x, (\phi_i^{v_h})'\right)\right)' &= 0 \quad \text{in } K_i \in K^h, \\ \phi_i^{v_h} &= \phi_i^0 \quad \text{on } \partial K_i = \{x_{i-1}, x_i\} . \end{aligned} \tag{23}$$

Consequently, linear multiscale basis functions can be used to represent $v_{\epsilon,h}$.

Multiscale numerical formulation. Our goal is to find $u_h \in W_h$ (consequently, $u_{\epsilon,h} (= E^{MsFEM} u_h) \in V_\epsilon^h$) such that

$$\langle A_{\epsilon,h} u_h, v_h \rangle = \int_\Omega f v_h dx , \quad \forall\, v_h \in W_h , \tag{24}$$

where

$$\langle A_{\epsilon,h} u_h, v_h \rangle = \sum_{K_i \in K^h} \int_{K_i} \left(a_\epsilon\left(x, u'_{\epsilon,h}\right)\right)' v'_h dx . \tag{25}$$

5. Analysis of multiscale finite element methods (MsFEM)

For the analysis of the MsFEM in our context, we recall that

$$a_\epsilon(x,\xi) = \frac{\xi}{1-\beta_\epsilon(x)|\xi|} \tag{26}$$

is a monotone function on $\xi \in \mathbb{R}$. Here, a satisfies, with $\xi \in \mathbb{R}, 0 \le |\xi| \le D < \dfrac{1}{\beta(x)} < 1$, the following properties (as in Lemma 3.1):

$$|a_\epsilon(x,\xi_1) - a_\epsilon(x,\xi_2)| \le C|\xi_1 - \xi_2|, \tag{27}$$

$$(a_\epsilon(x,\xi_1) - a_\epsilon(x,\xi_2))(\xi_1 - \xi_2) \ge |\xi_1 - \xi_2|^2, \tag{28}$$

where C is a positive constant defined in Lemma 3.1.

Recall that the inequalities (15) and (16) are the general conditions that guarantee the existence of a solution and are used in homogenization of nonlinear operator [14].

For the current periodic case, our goal is to show the convergence of the MsFEM in the limit as $\epsilon/h \to 0$. We consider $h = h(\epsilon)$ such that $h(\epsilon) \gg \epsilon$ and $h(\epsilon) \to 0$ as $\epsilon \to 0$.

Note that the homogenization of nonlinear partial differential equations has been studied previously (see [14], for instance). It can be shown that a sequence of solutions u_ϵ converges (up to a subsequence) to u in an appropriate norm, where $u \in W_0^{1,p}(\Omega)$ is a solution of the homogenized equation

$$-\left(a_*\left(x,u'\right)\right)' = f. \tag{29}$$

Next, we will present the convergence results for the MsFEM solutions in our context.

Theorem 5.1 ([13, 12])**.** *Given $a_\epsilon(x,\xi)$ in the form (26) as a periodic function with respect to x. Let u be a solution of the homogenized equation (29) and u_h be a MsFEM solution given by (24). Moreover, we assume that u_h' is uniformly bounded in $L^{2+\alpha}(\Omega)$ for some $\alpha > 0$. Then*

$$\lim_{\epsilon\to 0} \| u_h - u \|_{W_0^{1,2}(\Omega)} = 0, \tag{30}$$

where $h = h(\epsilon) \gg \epsilon$ and $h \to 0$ as $\epsilon \to 0$ (up to a subsequence).

Theorem 5.2 ([13, 12])**.** *Let u be a solution of the homogenized equation (29) and u_h be a MsFEM solution given by (24), with the coefficient $a_\epsilon(x,\xi) = a(x/\epsilon,\xi)$. Then*

$$\| u_h - u \|^2_{W_0^{1,2}(\Omega)} \le ch^2. \tag{31}$$

Proof. Based on the properties (27) and (28) of $a_\epsilon(x,\xi)$, the proofs of both Theorems 5.1 and 5.2 can be derived from the proofs of the corresponding theorems in [12, 13] in a similar manner. Since our problem is in one-dimensional setting, it follows that the boundaries of the coarse element consist of isolated points. Hence, the leading order resonance error proportional to ϵ/h caused by the linear boundary conditions as well as the second resonance error proportional to $(\epsilon/h)^2$ due to mismatch between the mesh size h and the small scale ϵ of the problem are canceled. □

6. Conclusions

In this paper, we investigate the analysis (including convergence) of the multiscale finite element method (MsFEM) for a periodically nonlinear elasticity problem in one-dimensional and strain-limiting settings. In particular, we obtained the convergence results for the MsFEM solutions in our context. An open question is extending this study to more general settings.

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